Optimization of Projectile Trajectory: A Review and Improvement to the Techniques

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Abstract— This paper is a review, reproduction and improvement to the techniques of optimization found in literature. The problem is to optimize an angle of launch so that the projectile may hit a fixed target in plane in minimum time. The problem is formulated as Augmented Lagrange (AL) equation and is solved numerically by Conjugate Gradient (CG) method. Golden Section Search (GSS) method is used for line search. An improvement of 23% to the computational efficiency is achieved by implementing improved algorithm of GSS *i.e.*, Golden Mean algorithm. By computing iterative values of Lagrange multiplier more efficiently an overall improvement of 38% is achieved.

Keywords: Optimization, Projectile, Conjugate Gradient, Golden Section search

1 Introduction

In this paper the optimum angle to launch a projectile is calculated, so that minimum time to hit the target is achieved. This is basically a projectile problem discussed in [1] that can be solved for determination of a point mass projectile trajectory for a specific angle and for different values of time. Since the aim is to minimize the time required to reach the target we need to calculate optimum angle of launch. From basic laws of projectile motion, we have

$$x_f = v_o \cos(\theta)\, t_f \quad (1)$$

$$y_f = v_0 \sin(\theta) - \frac{1}{2} g t_f^2 \quad (2)$$

where x_f and y_f are target position coordinates, v_o is initial velocity, g is force of gravity, and t_f is the final time to reach the target when projectile is launched at angle θ. Solving Eq. (1) and Eq. (2) we get two trajectories; one with angle 73.07° and other with 80.36°. See Fig. 1. When angle is not known and we want minimum time to reach the target, it becomes an optimization problem.

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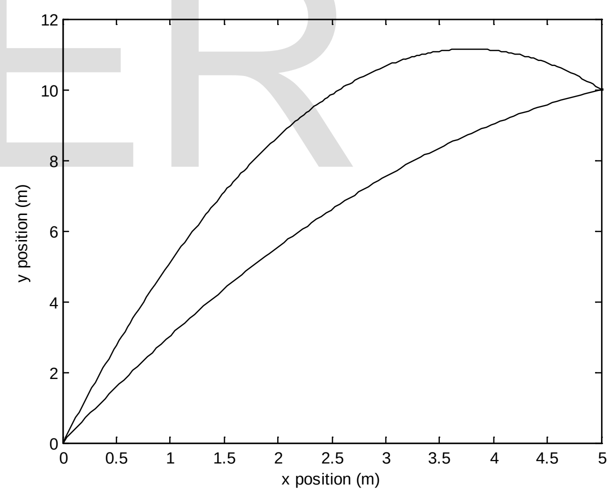

Figure 1. Trajectory plot of projectile launched at two different angles.

2 The Technique

First the problem is converted into constrained optimization problem. Problem formulation is based on Augmented Lagrange (AL) Method [2] and then it is solved using numerical techniques, *i.e.*, Conjugate Gradient (CG) Optimization technique [3] and Golden Section Search (GSS) [4].

2.1 Optimization Problem formulation

From (1):

$$C(\theta) = t_f = \frac{x_f}{v_0 \cos(\theta)} \tag{3}$$

(3) has θ as independent variable and t_f is the dependent variable. The cost function C(θ) is to be minimized to get minimum value of time in terms of optimal angle θ*. (2) and (3) are combined to give an equality constraint:

$$h(\theta) = y_f \cos^2(\theta) - x_f \sin(\theta)\cos(\theta) + \frac{g}{2}\left(\frac{x_f}{v_0}\right) = 0 \tag{4}$$

Optimization problem formulated with Lagrange method has a general form $\nabla C(\theta^*) + \Sigma\lambda_i \nabla \mathrm{h_i}(\theta^*) = 0$ where λ_i denotes i Langrange multipliers for i constraints. This problem is solved by Augmented Lagrange Method, in order to incorporate a penalty factor r_p, so that ill posed problem may give a close enough solution. Now the cost function is modified to

$$A(\theta) = C(\theta) + \lambda h(\theta) + r_p h^2(\theta) \tag{5}$$

where r_p is the Penalty factor associated with Augmented Lagrange Method

2.2 CG Algorithm

Algorithm we used is Polak-Ribiere variation to Fletcher-Reeves algorithm [5] for CG method.

2.2.1 Zeroth Iteration

2.2.1.1 Set initial values for λ, θ and r_p.

2.2.1.2 Calculate h(θ), A(θ).

2.2.1.3 Determine gradient of cost function A(θ) using equation

$$\nabla A(\theta) = \frac{A(\theta+\epsilon) - A(\theta)}{\epsilon} \tag{6}$$

where ϵ is a small increment in θ.

2.2.1.4 Determine search direction along θ by relation $p = -\nabla \mathrm{A}(\theta)$

2.2.2 Subsequent Iterations

2.2.2.1 To update value of θ find $\propto$ using 'Golden Section Search' method. Using equation

$$\theta_{k+1} = \theta_k + \alpha p_k \tag{7}$$

2.2.2.2 Update h(θ), $r_{p(k+1)} = 2r_p$, $\lambda_{k+1} = \lambda_k + 2r_p h(\theta)$ and repeat steps 2 and 3.

2.2.2.3 To update p, find $\beta = \frac{\nabla A_{k+1}(\nabla A_{k+1} - \nabla A_k)}{(\nabla A_k)^2}$ and use $p_{k+1} = -\nabla A_{k+1} + \beta_k p_k$

2.2.2.4 Repeat steps 2.2.2 to update θ untill $A(\theta)$ converges to some tolerance level (*i.e.*, the difference between two subsequent values of $A(\theta)$ should be less than 0.0001 in our case).

3 Improving The Techniques

The results obtained from given algorithm [1] took 13 iterations to converge as shown in table 1.

3.1 GSS Algorithm

GSS can be implemented using different approaches *i.e.*, Golden Mean, First Derivative and Parabolic Function algorithms [6]. Keeping all parameters unchanged, and trying Golden Mean [7] algorithm improved convergence of solution. See table 2.

3.2 Iterative Lagrange Multiplier Calculation

A further improvement is carried out by updating Lagrange multiplier after each iteration using more efficient formula. Formula used in [1] is misprinted and actual formula used is less efficient as can be seen from Table 1. and Table 2. Various other formulae are found in literature [8]. With formula mentioned in CG algorithm the solution converges after only 8 iterations. See Table 3.

4 Results

4.1 Discussion.

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Table 1 shows results given in [1]. Notice that A(θ) denotes to minimized time corresponding to optimal angle θ*. As shown in Table 2. the optimum time value has converged after 10 iterations (*i.e.*, the difference between two subsequent values of $A(\theta)$ should be less than 0.0001) as compared to 13 iterations, which shows more than 23% efficiency in computation. By improved formula for Lagrange multiplier update we get an overall 38% computational efficiency as shown in table 3. Note that subsequent results after convergence have not been shown in all three tables.

Table 1. Given values of launching angle (θ) and corresponding time A(θ) in seconds.

Iteration No.	Angle(θ)	A(θ) s
0	45	1.399
1	60.251	0.745
2	64.568	0.894
3	67.203	0.979
4	68.999	1.037
5	70.304	1.080
6	71.310	1.110
7	72.044	1.129
8	72.557	1.139
9	72.870	1.144
10	73.016	1.145
11	73.063	1.145
12	73.071	1.145
13	73.073	1.145

Table 2. Applying Golden Ratio Algorithm improves results by reducing number of iteration from 13 to 10

Iteration No.	Angle(θ)	A(θ)
0	45	1.399
1	64.64	0.898
2	67.24	0.981
3	69.0	1.03
4	70.31	1.08
5	71.30	1.11
6	72.04	1.13
7	72.56	1.140
8	72.87	1.143
9	73.02	1.144
10	73.06	1.145

Table 3. Using computational efficient updating method of Lagrange multiplier improves efficiency to 38%.

IterationNo.	Angle(θ)	A(θ)
0	45	1.399
1	66.44	0.966
2	67.28	1.035
3	70.47	1.092
4	71.32	1.115
5	72.03	1.132
6	72.56	1.141
7	72.90	1.143
8	73.04	1.145

4.2 Result Confirmation

Using Kuhn-Tucker Condition on optimal point the relation

$$\nabla C(\theta^*) = -\lambda^* \nabla h(\theta^*) \tag{8}$$

should hold. $\nabla C(\theta^*) = 3.684$, $\nabla h(\theta^*) = -1.4862$ and $\lambda^* = 2.4788$ which confirm that results are correct.

5 Conclusion

Techniques to optimize angle of launch for a point mass projectile to hit a fixed target in 1D (plane) in minimum time is reviewed and results have been reproduced accurately. The problem is formulated as Augmented Lagrange equation and is solved numerically by Conjugate Gradient method. Golden Section Search method is used for line search. Results have been accurately matched with given results [1]. An improvement of 23% to the computational efficiency is achieved by implementing improved algorithm of Golden Section Search. By computing iterative values of Lagrange multiplier more efficiently an overall improvement of 38% is achieved. Results are verified by applying Kuhn-Tucker condition on optimal point. This research can be extended to 2D intercept (Rendevouz) problem [9] and optimizing projectile angle incorporating air drag [10].

6 Nomenclature

α denotes line search parameter for step size.

β is linear combination of conjugate vector

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